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On the distribution of lattice points on spheres and level surfaces of polynomials ☆

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Abstract

The irregularities of distribution of lattice points on spheres and on level surfaces of polynomials are measured in terms of the discrepancy with respect to caps. It is found that the discrepancy depends on diophantine properties of the direction of the cap. If the direction of the cap is diophantine, in case of the spheres, close to optimal upper bounds are found. The estimates are based on a precise description of the Fourier transform of the set of lattice points on polynomial surfaces.

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1. Introduction

The uniformity of the distribution of lattice points on spheres has been extensively studied and proved in dimension at least 4 , see [P,GF], and later in dimension 3 [D] using difficult estimates for the Fourier coefficients of modular forms.

Here we study the discrepancy on spheres, and more generally on level surfaces of certain positive homogeneous polynomials, with respect to caps, which are intersections of the surface with half-spaces.

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To describe our results first in case of spheres, let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, and for $\lambda \in \mathbb{N}$ let Z_λ be the set of lattice points of length $\lambda^{1/2}$ projected to the unit sphere: $Z_\lambda = \{\lambda^{-1/2}m\colon\ m \in \mathbb{Z}^n,\ |m|^2 = \lambda\}$. Here $|m| = (m_1^2 + \cdots + m_n^2)^{1/2}$ denotes the Euclidean length. Let $N_\lambda = |Z_\lambda|$ be the number of lattice points of length $\lambda^{1/2}$.

For given $0 \leqslant a < 1$ and a unit vector ξ define the spherical cap $C_{a,\xi} = \{x \in S^{n-1}\colon\ x \cdot \xi \geqslant a\}$, and the corresponding *discrepancy* as the difference between the actual and the expected number of points of Z_λ which lie on the cap $C_{a,\xi}$:

$$D_n(\xi, \lambda) = |Z_\lambda \cap C_{a,\xi}| - N_\lambda \sigma(C_{a,\xi}), \tag{1.1}$$

where σ denotes the normalized surface area measure on S^{n-1}. Our aim is to prove upper bounds for the discrepancy when the direction of the cap ξ satisfies certain diophantine conditions, which we describe below.

A point $\alpha \in \mathbb{R}^{n-1}$ is called *diophantine* if for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that for all $q \in \mathbb{N}$

$$\|q\alpha\| = \min_{m \in \mathbb{Z}^{n-1}} |q\alpha - m| \geqslant C_\varepsilon q^{-\frac{1}{n-1} - \varepsilon}. \tag{1.2}$$

Correspondingly a point $\xi \in S^{n-1}$ is called *diophantine*, if for every $1 \leqslant i \leqslant n$ for which $\xi_i \neq 0$, the point $\alpha^i \in \mathbb{R}^{n-1}$ is diophantine, where the coordinates of α^i are obtained by dividing each coordinate of ξ by ξ_i and deleting the ith coordinate. It is not hard to show, see the next section, that the complement of diophantine points has measure 0 in $\mathbb{R}^{n-1}$ and hence in S^{n-1} as well.

Theorem 1. *Let $n \geqslant 4$ and let $\xi \in S^{n-1}$ be a diophantine point. Then for every $\varepsilon > 0$, one has*

$$\bigl|D_n(\xi, \lambda)\bigr| \leqslant C_{\xi,\varepsilon} \lambda^{\frac{n-1}{4} + \varepsilon}. \tag{1.3}$$

We note that for $n \geqslant 4$, and if $n = 4$ assuming that 4 does not divide λ, one has that $N_\lambda \gtrsim \lambda^{\frac{n}{2} - 1}$, thus (1.3) implies

$$\bigl|D_n(\xi, \lambda)\bigr| \leqslant C_\varepsilon N_\lambda^{\frac{1}{2} + \frac{1}{2(n-2)} + \varepsilon}. \tag{1.4}$$

On the other hand, it is known that for any set of N points on the unit sphere S^{n-1} the L^2 average of the discrepancy is at least: $N^{\frac{1}{2} - \frac{1}{2(n-1)}}$, see [Be,M2]. Thus our estimates are asymptotically sharp as $n \to \infty$.

Also, such estimates are not possible, in high dimensions, without some restrictions on the direction ξ. Indeed, if $\xi = (0, \ldots, 0, 1)$ then the boundary of the cap $C_{a,\xi}$ can contain as many as $\lambda^{\frac{n-3}{2}} \approx N_\lambda^{1 - \frac{1}{n-2}}$ lattice points for certain values of a. Thus the *discrepancy* must change by this amount at such values of a.

In low dimension when $n = 4$, the best previous estimate for the normalized discrepancy $D(\xi, \lambda)/N_\lambda$ was given in [GF] of the order of $\lambda^{-1/5+\varepsilon}$ while we get the improvement $\lambda^{-1/4+\varepsilon}$. In case $n = 4$ and $\lambda = 4^k$ there are only 24 lattice points of length $\lambda^{1/2}$, estimates for the discrepancy become trivial in such degenerate cases.

Next we describe similar estimates in case where spheres are replaced by level surfaces of positive homogeneous polynomials. Let $p(m) = p(m_1, \dots, m_n)$ be a positive homogeneous integral polynomial of degree d. Let S_p be the unit level surface of the polynomial p, and let $\sigma_p = c_p \frac{dS_p}{|\nabla p|}$ where dS_p denote the surface-area measure on S_p and ∇p stands for the gradient of p. The constant $c_p > 0$ is chosen to have total measure 1. For $a > 0$ and a unit vector ξ, define the cap $C_{a,\xi} = \{x \in S_p\colon a \leqslant x \cdot \xi\}$ as before.

For a positive integer λ let $Z_{p,\lambda} = \{\lambda^{-1/d} m\colon m \in \mathbb{Z}^n,\ p(m) = \lambda\}$, $N_{p,\lambda} = |Z_{p,\lambda}|$, and define the discrepancy by

$$D_p(\xi, \lambda) = |Z_{p,\lambda} \cap C_{a,\xi}| - N_{p,\lambda}\sigma_p(C_{a,\xi}). \tag{1.5}$$

To ensure that there are enough many solutions of the diophantine equation $p(m) = \lambda$, we assume that $p(z)$ is *non-singular*, that is $\nabla p(z) \neq 0$ for $z \in \mathbb{C}^n$, $z \neq 0$. Indeed this condition excludes polynomials like $p(z) = z_1^d$ or $p(z) = (z_1^2 + \dots + z_n^2)^{d/2}$. It is implicit in earlier works on the Hardy–Littlewood method of exponential sums and shown in [M1], that if $n > (d-1)2^d$ and p is non-singular, then there is an infinite arithmetic progression Λ, such that for each $\lambda \in \Lambda$ one has $N_{p,\lambda} \gtrsim \lambda^{\frac{n}{d}-1}$. We will refer to such a set Λ as a set of *regular values* of p.

Theorem 2. *Let $n > (d-1)2^d$, and let $p(m)\colon \mathbb{Z}^n \to \mathbb{Z}$ be a non-singular, positive, homogeneous, integral polynomial of degree d. If $\xi \in S^{n-1}$ is a diophantine point, then there is a $\eta > 0$ depending only on the dimension n and the degree d, such that*

$$\left|D_p(\xi, \lambda)\right| \leqslant C_{p,\xi} \lambda^{\frac{n}{d}-1-\eta}. \tag{1.6}$$

If $\lambda \in \Lambda$ is a regular value of p, then (1.6) implies

$$\left|D_p(\xi, \lambda)\right| \leqslant C_{p,\xi} N_{p,\lambda}^{1-\eta'}$$

again with a constant $\eta > 0$ depending only on n and d. Let us remark that assuming the stronger condition: $n > (d-1)2^{d+1}$ one can take $\eta' = \frac{1}{(d-1)2^d}$ which depends only on the degree d. However we do not pursue such estimates here, as it would require to rework some of the error estimates in [M1] and would greatly increase the length of the paper.

For $\lambda \in \mathbb{N}$ let $\omega_{p,\lambda}\colon \mathbb{Z}^n \to \{0, 1\}$ be the indicator function of the solution set $p(m) = \lambda$. Then both estimates (1.3) and (1.6) are based on an asymptotic formula for its Fourier transform

$$\hat{\omega}_{p,\lambda}(\xi) = \sum_{m \in \mathbb{Z}^n,\ p(m) = \lambda} e^{-2\pi i m \cdot \xi}. \tag{1.7}$$

In case spheres, when $p(m) = \sum_{i=1}^n m_i^2$, such a formula was derived in [MSW] (see Proposition 4.1) for $n \geqslant 5$. Here we will introduce the so-called Kloostermann refinement to include the case $n = 4$ and to obtain a better error term.

Lemma 1. *Let $n \geqslant 4$. Then one has*

$$\hat{\omega}_\lambda(\xi) = \gamma_n \lambda^{\frac{n}{2}-1} \sum_{q \leqslant \lambda^{1/2}} m_{q,\lambda}(\xi) + \mathcal{E}_\lambda(\xi), \tag{1.8}$$

where

$$\left|\mathcal{E}_\lambda(\xi)\right| \leqslant C_\varepsilon \lambda^{\frac{n-1}{4}+\varepsilon} \tag{1.9}$$

holds uniformly in ξ *for every* $\varepsilon > 0$. *Moreover*

$$m_{q,\lambda}(\xi) = \sum_{l\in\mathbb{Z}^n} K(q,l,\lambda)\psi(q\xi - l)\tilde{\sigma}\bigl(\lambda^{\frac{1}{2}}(\xi - l/q)\bigr), \tag{1.10}$$

where

$$K(q,l,\lambda) = q^{-n} \sum_{(a,q)=1} \sum_{s\in(\mathbb{Z}/q\mathbb{Z})^n} e^{2\pi i \frac{a(|s|^2-\lambda)+s\cdot l}{q}} \tag{1.11}$$

$\tilde{\sigma}$ *denotes the Fourier transform of the measure* σ *on* S^{n-1}, *and* ψ *is a smooth cut-off function supported on* $\max_j |\xi_j| \leqslant 1/4$ *and constant* 1 *on* $\max_j |\xi_j| \leqslant 1/8$. *Moreover one has the bounds*

$$\left|\tilde{\sigma}(\xi)\right| \leqslant C\bigl(1+|\xi|\bigr)^{-\frac{n-1}{2}}, \tag{1.12}$$

$$\left|K(q,l,\lambda)\right| \leqslant C_\varepsilon q^{-\frac{n-1}{2}+\varepsilon}(\lambda,q_1)^{\frac{1}{2}}2^{\frac{r}{2}}, \tag{1.13}$$

where $q = q_1 2^r$ *with* q_1 *odd, and* (λ, q_1) *denotes the greatest common divisor of* λ *and* q_1.

We remark that (1.12) is a standard stationary phase estimate, and (1.13) follows from Weil's estimate and the multiplicative properties of Kloostermann sums, see Section 4.

The factor $(\lambda, q_1)^{1/2}2^{r/2}$ is of size λ^ε on average, in fact one has the estimate

$$\sum_{q\leqslant\lambda^{1/2}} q^\beta(\lambda,q_1)^{\frac{1}{2}}2^{\frac{r}{2}} \leqslant C_{\beta,\varepsilon}\lambda^{\frac{\beta+1}{2}+\varepsilon}. \tag{1.14}$$

The organization of the paper is as follows. In Section 2 we establish some basic properties of diophantine points. Most of these are known but for the sake of completeness we include their proofs. In Section 3 we prove estimate (1.3) assuming Lemma 1. This will be shown in Section 4, using a general form of the Kloostermann refinement proved in [H-B]. Estimate (1.6) on the upper bound of the discrepancy for polynomial surfaces will be shown in Section 5. The proof is essentially the same as in case of spheres, based on asymptotic formula (0.6) proved in [M1], analogues to (1.8).

2. Some properties of diophantine points

Let us call a points $\alpha \in \mathbb{R}^n$ of *type* ε if it satisfies (1.2) with a given $\varepsilon > 0$.

Proposition 1. *For every* $\varepsilon > 0$ *the set of points* $\alpha \in [0,1]^{n-1}$ *of type* ε *has measure* 1.

Proof. If a point α is not of "type ε" then there are infinitely many q's such that: $\|q\xi\| \geqslant q^{-\frac{1}{n-1}-\varepsilon}$. This means there exists $m \in \mathbf{Z}^n$ such that: $|\xi - m/q| \leqslant q^{-\frac{n}{n-1}-\varepsilon}$. However the sum of the volume of such neighborhoods around the points $m/q \in [0,1]^{n-1}$ is bounded by $q^{-1-\varepsilon}$. Therefore the set of points which belong to infinitely many of such neighborhoods has measure 0. $\square$

This shows that the set of points $\alpha \in \mathbb{R}^{n-1}$ which are not diophantine has measure 0. Indeed α is diophantine if it is of type $\varepsilon_k = (1/2)^k$ for $k = 1, 2, \ldots$, and in that case $\alpha + m$ is also diophantine for every $m \in \mathbb{Z}^n$. Next we show that $\|q\alpha\| \approx 1$ on average if α is diophantine.

Proposition 2. *Let $\alpha \in [0,1]^{n-1}$ be diophantine, $Q > 1$ and $0 < k < n-1$. Then for every $\varepsilon > 0$ one has*

$$\sum_{q \leqslant Q} \|q\alpha\|^{-k} \leqslant C_\varepsilon Q^{1+\varepsilon}. \tag{2.1}$$

Proof. Let $\varepsilon > 0$. Consider the set of points $\{q\alpha\} \in [-1/2, 1/2]^{n-1}$, $1 \leqslant q \leqslant Q$, where $\{q\alpha\} = q\alpha - [q\alpha]$ and $[q\alpha]$ denotes the closest lattice points to $q\alpha$. If $q_1 \neq q_2$ then

$$\left|\{q_1\alpha\} - \{q_2\alpha\}\right| \geqslant \left\|(q_1 - q_2)\alpha\right\| \geqslant C_\varepsilon Q^{-\frac{1}{n-1} - \frac{\varepsilon}{n}}. \tag{2.2}$$

Thus the number of points in a dyadic annulus $2^{-j} \leqslant \|q\alpha\| < 2^{-j+1}$ is bounded by $2^{-(n-1)j} Q^{1+\varepsilon}$ and the sum in (2.1) is convergent for $k < n-1$. $\square$

Proposition 3. *Let $\xi \in S^{n-1}$ be diophantine, and assume that $\max_j |\xi_j| = |\xi_n|$. Let $t \geqslant 1$, $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$, $\alpha_j = \xi_j/\xi_n$ and $q = [t\xi_n]$. Then one has*

$$\|t\xi\| \geqslant \frac{1}{n} \|q\alpha\|. \tag{2.3}$$

Proof. Note that

$$t\xi_j = t\xi_n \alpha_j = [t\xi_n]\alpha_j \pm \|t\xi_n\| \alpha_j$$

thus

$$|q\alpha_j - m_j| \leqslant |t\xi_j - m_j| + \|t\xi_n\|$$

thus taking $m_j = [t\xi_j]$ one has

$$\|q\alpha_j\| \leqslant \|t\xi_j\| + \|t\xi_n\|$$

and summing for $1 \leqslant j \leqslant n-1$ gives (2.3). $\square$

Lemma 2. *Suppose $\xi \in S^{n-1}$ is diophantine. For $t \geqslant 1$ and $T \geqslant 1$, one has for every $\varepsilon > 0$*

$$\|t\xi\| \geqslant C_\varepsilon t^{-\frac{1}{n-1} - \varepsilon}, \tag{2.4}$$

$$\int_1^T \|t\xi\|^{-k} \leqslant C_\varepsilon T^{1+\varepsilon} \tag{2.5}$$

for $0 < k < n-1$.

Proof. By permuting the coordinates of ξ (which does not affect the property of being diophantine), one can assume that $\max_j |\xi_j| = |\xi_n|$. Inequality (2.4) follows immediately from (2.3) and the definition of a diophantine point. Similarly (2.5) is reduced to (2.1) by observing that for a fixed q, the set of t's for which $q = [t\xi_n]$ is an interval of length at most $1/\xi_n \leqslant \sqrt{n}$. $\square$

3. Upper bounds for the discrepancy

If χ_a denotes the indicator function of the interval $[a, 1 + a]$, then the discrepancy may be written as

$$D_n(\xi, \lambda) = \sum_{|m|^2=\lambda} \chi_a\bigl(\lambda^{-\frac{1}{2}} m \cdot \xi\bigr) - N_\lambda \int_{S^{n-1}} \chi_a(x \cdot \xi)\, d\sigma(x). \tag{3.1}$$

The function χ_a can be replaced with a smooth function $\phi_{a,\delta}$ by making a small error in the discrepancy. Indeed, let $0 \leqslant \phi(t) \leqslant 1$ be smooth function supported in $[-1, 1]^n$, such that $\int \phi = 1$. Let $\phi^{\pm}_{a,\delta} = \chi_{a\pm\delta} * \phi_\delta$, where $\phi_\delta(t) = \delta^{-1}\phi(t\delta^{-1})$ and define the *smoothed* discrepancy as

$$D_n\bigl(\phi^{\pm}_{a,\delta}, \xi, \lambda\bigr) = \sum_{|m|^2=\lambda} \phi^{\pm}_{a,\delta}\bigl(\lambda^{-\frac{1}{2}} m \cdot \xi\bigr) - N_\lambda \int_{S^{n-1}} \phi^{\pm}_{a,\delta}(x \cdot \xi)\, d\sigma(x). \tag{3.2}$$

Proposition 4. *One has*

$$\bigl|D_n(\xi, \lambda)\bigr| \leqslant \max\bigl(\bigl|D_n\bigl(\phi^{+}_{a,\delta}, \xi, \lambda\bigr)\bigr|, \bigl|D_n\bigl(\phi^{-}_{a,\delta}, \xi, \lambda\bigr)\bigr|\bigr) + C_n \delta N_\lambda. \tag{3.3}$$

Proof. Note that $\phi^{-}_{a,\delta}(t) \leqslant \chi_a(t) \leqslant \phi^{+}_{a,\delta}(t)$ thus

$$\sum_{|m|^2=\lambda} \phi^{-}_{a,\delta}\bigl(\lambda^{-\frac{1}{2}} m \cdot \xi\bigr) \leqslant \sum_{|m|^2=\lambda} \chi_a\bigl(\lambda^{-\frac{1}{2}} m \cdot \xi\bigr) \leqslant \sum_{|m|^2=\lambda} \phi^{+}_{a,\delta}\bigl(\lambda^{-\frac{1}{2}} m \cdot \xi\bigr)$$

and

$$N_\lambda \int_{S^{n-1}} \phi^{+}_{a,\delta}(x \cdot \xi)\, d\sigma(x) \geqslant N_\lambda \int_{S^{n-1}} \chi_a(x \cdot \xi)\, d\sigma(x) \geqslant N_\lambda \int_{S^{n-1}} \phi^{-}_{a,\delta}(x \cdot \xi)\, d\sigma(x).$$

Subtracting the above inequalities, (3.3) follows from

$$\int_{S^{n-1}} \bigl(\phi^{+}_{a,\delta} - \phi^{-}_{a,\delta}\bigr)(x \cdot \xi)\, d\sigma(x) \leqslant C_n \delta. \qquad \square$$

In what follows, we take $\delta = \lambda^{-n}$ and write $\phi_{a,\delta}$ for $\phi^{\pm}_{a,\delta}$, as our estimates work the same way for both choices of the sign. By taking the inverse Fourier transform of $\phi_{a,\delta}(t)$ one has

$$\sum_{|m|^2=\lambda} \phi_{a,\delta}\big(\lambda^{-\frac{1}{2}} m\cdot\xi\big) = \int_{\mathbb{R}} \lambda^{\frac{1}{2}} \hat{\phi}_{a,\delta}\big(t\lambda^{\frac{1}{2}}\big)\hat{\omega}_\lambda(t\xi)\,dt, \tag{3.4}$$

$$\int_{S^{n-1}} \phi_{a,\delta}(x\cdot\xi)\,d\sigma(x) = \int_{\mathbb{R}} \hat{\phi}_{a,\delta}(t)\tilde{\sigma}(t\xi)\,dt. \tag{3.5}$$

We substitute the asymptotic formula (1.8) into (3.4) and study the contribution of each term separately:

$$I_{q,\lambda} = \int_{\mathbb{R}} \lambda^{\frac{1}{2}} \hat{\phi}_{a,\delta}\big(t\lambda^{\frac{1}{2}}\big) m_{q,\lambda}(t\xi)\,dt, \tag{3.6}$$

$$E_\lambda = \int_{\mathbb{R}} \lambda^{\frac{1}{2}} \hat{\phi}_{a,\delta}\big(t\lambda^{\frac{1}{2}}\big)\mathcal{E}_\lambda(t\xi)\,dt. \tag{3.7}$$

To estimate the error term in (3.7) note that

$$\int_{\mathbb{R}} \lambda^{\frac{1}{2}} \big|\hat{\phi}_{a,\delta}\big(t\lambda^{\frac{1}{2}}\big)\big|\,dt \leqslant C \int_{\mathbb{R}} \big(1+|t|\big)^{-1}\big(1+\delta|t|\big)^{-1} \leqslant C\log\lambda.$$

Thus by (1.9) one has for every $\varepsilon > 0$

$$|E_\lambda| \leqslant C_\varepsilon \lambda^{\frac{n-1}{4}+\varepsilon}. \tag{3.8}$$

Next, decompose the range of integration in (3.6) as

$$I_{q,\lambda} = \int_{|t|<1/8q} + \int_{|t|\geqslant 1/8q} = I^1_{q,\lambda} + I^2_{q,\lambda}. \tag{3.9}$$

A crucial point is that if $|t| < 1/8q$ then $\psi(q\xi - l) = 0$ unless $l = 0$ moreover $\psi(tq\xi) = 1$ since $|tq\xi_j| < 1/8q$ for each j, hence

$$m_{q,\lambda}(t\xi) = K(q,0,\lambda)\tilde{\sigma}\big(\lambda^{\frac{1}{2}} t\xi\big).$$

Thus by (3.6) and a change of variables: $t := t\lambda^{1/2}$

$$I^1_{q,\lambda} = K(q,l,\lambda) \int_{|t|<\lambda^{1/2}/8q} \hat{\phi}_{a,\delta}(t)\tilde{\sigma}(t\xi)\,dt. \tag{3.10}$$

Proposition 5. *One has for every $\varepsilon > 0$*

$$\Bigg|\gamma_n \lambda^{\frac{n}{2}-1} \sum_{q\leqslant\lambda^{1/2}} I^1_{q,\lambda} - N_\lambda \int_{S^{n-1}} \phi_{a,\delta}(x\cdot\xi)\,d\sigma(x)\Bigg| \leqslant C_\varepsilon \lambda^{\frac{n-1}{4}+\varepsilon}. \tag{3.11}$$

Proof. Using (1.12), one has

$$\int\limits_{|t|\geqslant\lambda^{1/2}/8q} \left|\hat{\phi}_{a,\delta}(t)\tilde{\sigma}(t\xi)\right| dt \leqslant C_\varepsilon \lambda^{-\frac{n-1}{4}+\varepsilon} q^{\frac{n-1}{2}}. \tag{3.12}$$

Thus by (3.5) and (3.10)

$$\left| I^1_{q,\lambda} - K(q,0,\lambda) \int\limits_{S^{n-1}} \phi_{a,\delta}(x\cdot\xi)\, d\sigma(x)\right| \leqslant C_\varepsilon \lambda^{-\frac{n-1}{4}+\varepsilon} q^{\frac{n-1}{2}} \left|K(q,0,\lambda)\right|.$$

Substituting $\xi = 0$ in (1.8) one has

$$\left| N_\lambda - \gamma_n \lambda^{\frac{n}{2}-1} \sum_{q\leqslant\lambda^{1/2}} K(q,0,\lambda)\right| \leqslant C_\varepsilon \lambda^{\frac{n-1}{4}+\varepsilon}. \tag{3.13}$$

Using (1.13) and (1.14), the left side of (3.11) is estimated by

$$C_\varepsilon\left(\lambda^{\frac{n-1}{4}+\varepsilon} + \lambda^{\frac{n-3}{4}+\varepsilon} \sum_{q\leqslant\lambda^{1/2}} q^\varepsilon (\lambda, q_1)^{\frac{1}{2}} 2^{\frac{r}{2}}\right) \leqslant C_\varepsilon \lambda^{\frac{n-1}{4}+\varepsilon}. \qquad \square \tag{3.14}$$

Proposition 6. *Let $\xi \in S^{n-1}$ diophantine. Then for every $\varepsilon > 0$*

$$\sum_{q\leqslant\lambda^{1/2}} \left|I^2_{q,\lambda}\right| \leqslant C_{\xi,\varepsilon} \lambda^{-\frac{n-3}{4}+\varepsilon}. \tag{3.15}$$

Proof. First, note that $\psi(q\xi - l) = 0$ unless $l = [q\xi]$, that is the closest lattice point to the point $q\xi \in \mathbb{R}^n$. Using the notation $\{q\xi\} = q\xi - [q\xi]$ one may write

$$m_{q,\lambda}(t\xi) = K\big(q, [qt\xi], \lambda\big)\psi\big(\{qt\xi\}\big)\tilde{\sigma}\left(\frac{\lambda^{\frac{1}{2}}}{q}\{qt\xi\}\right). \tag{3.16}$$

By making a change of variables $t := qt$, it follows from estimates (1.12) and (1.13)

$$\left|I^2_{q,\lambda}\right| \leqslant C_\varepsilon \big(\lambda^{\frac{1}{2}}/q\big)^{-\frac{n-3}{2}} q^{-\frac{n-1}{2}+\varepsilon} (\lambda, q_1)^{\frac{1}{2}} 2^{\frac{r}{2}} J_\lambda, \tag{3.17}$$

where

$$J_\lambda = \int\limits_{|t|\geqslant 1/8} \left|\hat{\phi}_{a,\delta}\big(t\lambda^{\frac{1}{2}}/q\big)\right| \|t\xi\|^{-\frac{n-1}{2}}\, dt \tag{3.18}$$

and $\|t\xi\|$ denotes the distance of the point $t\xi$ to the nearest lattice point. For $q \leqslant \lambda^{1/2}$ one has

$$\left|\hat{\phi}_{a,\delta}\big(t\lambda^{\frac{1}{2}}/q\big)\right| \leqslant C\big(\lambda^{\frac{1}{2}}/q\big)^{-1} |t|^{-1} \big(1 + \delta|t|\big)^{-1}. \tag{3.19}$$

To estimate the integral J_λ one uses (2.5) and integrates over dyadic intervals $2^j \leqslant |t| < 2^{j+1}$ ($j \geqslant -3$). For a fixed j one obtains

$$\int_{2^j}^{2^{j+1}} t^{-1}(1+\delta t)^{-1}\|t\xi\|^{-\frac{n-1}{2}}\,dt \leqslant C_\varepsilon 2^{j\varepsilon}\left(1+\delta 2^j\right)^{-1}. \tag{3.20}$$

Summing over j this gives: $J_\lambda \leqslant C_\varepsilon(\lambda^{1/2}/q)^{-1}\lambda^\varepsilon$. Substituting into (3.18)

$$\left|I^2_{q,\lambda}\right| \leqslant C_\varepsilon \lambda^{-\frac{n-1}{4}+\varepsilon} q^\varepsilon (\lambda, q_1)^{\frac{1}{2}} 2^{\frac{r}{2}}. \tag{3.21}$$

Summing over $q \leqslant \lambda^{1/2}$, and using (1.14), the proposition follows. □

Theorem 1 follows immediately from Propositions 4–6, and estimate (3.8).

4. Fourier transform of the set of lattice points on spheres

In this section we prove the asymptotic formula (1.8). The analysis is very similar to that of [MSW], except we use a general form of the so-called Kloostermann refinement [H-B, Lemma 7], to be described below.

Theorem *(Heath-Brown). Let $Q(m)$ be a polynomial with integral coefficients, λ, N are natural numbers and $w(x)$ is a non-negative, bounded function. Then one has*

$$\sum_{Q(m)=\lambda} w(m) = \sum_{q\leqslant N} \int_{-1/(qN)}^{1/(qN)} e^{-2\pi i\lambda\tau} S_0(q,\tau)\,d\tau + E_1(\lambda), \tag{4.1}$$

where

$$\left|E_1(\lambda)\right| \leqslant CN^{-2}\sum_{q\leqslant N}\sum_{|u|\leqslant q/2}\left(1+|u|\right)^{-1}\max_{\tau\approx 1/(qN)}\left|S_u(q,\tau)\right|. \tag{4.2}$$

Here $C>0$ is an absolute constant and

$$S_u(q,\tau) = \sum_{(a,q)=1} e^{2\pi i\frac{\bar{a}u-a\lambda}{q}} S(a/q+\tau), \qquad S(\alpha) = \sum_{m\in\mathbb{Z}^n} e^{2\pi i\alpha Q(m)} w(m). \tag{4.3}$$

Note that the original formulation of Lemma 7 in [H-B] is for the homogeneous equation $F(m)=0$, which can be used for the equation $Q(m)=\lambda$ by choosing: $F(m)=Q(m)-\lambda$.

We will apply the above result to the polynomial $Q(m)=|m|^2$ and choose $N=[\lambda^{1/2}]$, $\delta=\lambda^{-1}$ and $w(x)=e^{-2\pi\delta|x|^2}e^{2\pi i x\cdot\xi}$, for given $\lambda\in\mathbb{N}$ and $\xi\in\mathbb{R}^n$. Note that

$$\sum_{Q(m)=\lambda} w(m) = \sum_{|m|^2=\lambda} e^{-2\pi\delta|m|^2}e^{2\pi i m\cdot\xi} = e^{-2\pi}\hat{\omega}_\lambda(\xi).$$

Substituting into (4.3) with $\alpha = a/q + \tau$ one obtains

$$S(a/q+\tau) = \sum_{m\in\mathbb{Z}^n} e^{2\pi i \frac{a}{q}|m|^2} h_{\tau,\delta}(m) \tag{4.4}$$

with $h_{\tau,\delta}(x) = e^{2\pi i((\tau+i\delta)|x|^2+x\cdot\xi)}$. Writing $m = qm_1 + s$ where $m_1 \in \mathbb{Z}^n$ and $s \in (\mathbb{Z}/q\mathbb{Z})^n$ and applying Poisson summation in m_1 one obtains

$$\begin{aligned} S(a/q+\tau) &= \sum_{s\in(\mathbb{Z}/q\mathbb{Z})^n} e^{2\pi i \frac{a}{q}|s|^2} h_{\tau,\delta}(qm_1+s) \\ &= \sum_{l\in\mathbb{Z}^n} G(a,l,q)\tilde{h}_{\tau,\delta}(\xi - l/q), \end{aligned} \tag{4.5}$$

where $G(a,l,q)$ is a standard normalized Gaussian sum, satisfying the basic estimate

$$\left|G(a,l,q)\right| = q^{-n}\left|\sum_{s\in(\mathbb{Z}/q\mathbb{Z})^n} e^{2\pi i \frac{a|s|^2 - l\cdot s}{q}}\right| \leqslant Cq^{-\frac{n}{2}}. \tag{4.6}$$

The function $\tilde{h}_{\tau,\delta}$ denotes the Fourier transform of $h_{\tau,\delta}$ on $\mathbb{R}^n$, which can be evaluated explicitly

$$\tilde{h}_{\tau,\delta}(\xi - l/q) = \big(2(\tau+i\delta)\big)^{-\frac{n}{2}} e^{-\frac{\pi|q\xi - l|^2}{2q^2(\delta - i\tau)}}. \tag{4.7}$$

On the range when $|\tau| \approx 1/qN \approx 1/q\lambda^{1/2}$, one has $\mathrm{Re}(\frac{1}{q^2(\delta-i\tau)}) \geqslant c$ for some absolute constant $c > 0$. Thus one has

$$\left|\tilde{h}_{\tau,\delta}(\xi - l/q)\right| \leqslant Cq^{\frac{n}{2}}\lambda^{\frac{n}{4}} e^{-c|q\xi - l|^2}. \tag{4.8}$$

Also, from (4.5)

$$S_u(q,\tau) = \sum_{l\in\mathbf{Z}^n} K(q,l,\lambda;u)\tilde{h}_{\tau,\delta}(\xi - l/q), \tag{4.9}$$

where

$$K(q,l,\lambda;u) = \sum_{(a,q)=1} e^{2\pi i \frac{\bar{a}u - a\lambda}{q}} G(a,l,q). \tag{4.10}$$

Next, we derive estimates (1.13) and (1.14). Variants of these estimates are known in the literature and are going back to the original work of Kloostermann. However as it is hard to quote the exact results needed here, we include their proofs.

Proposition 7. *Let $K(q,l,\lambda;u)$ be the exponential sum defined in* (4.10). *Then for every $\varepsilon > 0$, one has*

$$\left|K(q,l,\lambda;u)\right| \leqslant C_\varepsilon q^{\frac{n-1}{2}+\varepsilon}(\lambda,q_1)^{\frac{1}{2}}2^{\frac{r}{2}},$$

where $q = q_1 2^r$ with q_1 odd, and (λ, q_1) denotes the greatest common divisor of λ and q_1.

Proof. It is immediate from (4.6) that

$$\left|K(q,l,\lambda;u)\right| \leqslant cq^{-\frac{n}{2}+1}. \tag{4.11}$$

The Gaussian sum given in (1.6) is a product of one-dimensional sums. For q odd, by completing the square in the exponent, it may be written in the form (see also [S, Chapter 4])

$$G(a,l,q) = q^{-n} \varepsilon_q^n \left(\frac{q}{a}\right)^n e^{-2\pi i \frac{\bar{4}\bar{a}|l|^2}{q}} G(1,0,q)^n,$$

where $(\frac{q}{a})$ denotes the Jacobi symbol, ε_q is a 4th root of unity, and $\bar{a}$ denotes the multiplicative inverse of a mod q. Substituting this into (4.10) one obtains

$$K(q,l,\lambda;u) = \varepsilon_q^n q^{-n} G(1,0,q)^n \sum_{(a,q)=1} \left(\frac{q}{a}\right)^n e^{2\pi i \frac{a\lambda + \bar{4}\bar{a}(u-|l|^2)}{q}}. \tag{4.12}$$

The sum in (4.12) is a Kloostermann sum or Salie sum depending on whether n is even or odd. Weil's estimates [S, Chapter 4] imply

$$\left|K(q,l,\lambda;u)\right| \leqslant C_\varepsilon q^{-\frac{n-1}{2}+\varepsilon} (\lambda,q)^{\frac{1}{2}}. \tag{4.13}$$

Estimate (1.13) follows by writing $q = q_1 q_2$, with q_1 odd and $q_2 = 2^r$, applying (4.13) to q_1, (4.11) to $q_2 = 2^r$ and using the multiplicative property

$$K(q,l,\lambda;u) = K\big(q_1, l\bar{q}_2, \lambda; u\bar{q}_2^2\big) K\big(q_2, l\bar{q}_1, \lambda; u\bar{q}_1^2\big), \tag{4.14}$$

where $q_1\bar{q}_1 \equiv 1 \pmod{q_2}$, and $q_2\bar{q}_2 \equiv 1 \pmod{q_1}$. Property (4.14) is well known, and is an easy computation using the Chinese Remainder Theorem. □

Proposition 8. *Let $\beta \in \mathbb{R}$. Then for every $\varepsilon > 0$, one has*

$$\sum_{q \leqslant \lambda^{1/2}} q^\beta (\lambda, q_1)^{\frac{1}{2}} 2^{\frac{r}{2}} \leqslant C_{\beta,\varepsilon} \lambda^{\frac{\beta+1}{2}+\varepsilon}.$$

Proof. Let $1 \leqslant \mu \leqslant \lambda^{1/2}$. First we show that

$$\sum_{q \leqslant \mu} (\lambda, q_1)^{\frac{1}{2}} 2^{\frac{r}{2}} \leqslant C_\varepsilon \lambda^\varepsilon \mu. \tag{4.15}$$

To see this, write $d = (\lambda, q_1)$ and $q_1 = dt$. Then d divides λ and $d2^r t \leqslant \mu$, hence the left side of (4.15) is majorized by

$$\sum_{d|\lambda} \sum_{r \in \mathbb{N}} d^{\frac{1}{2}} 2^{\frac{r}{2}} \frac{\mu}{d2^r} \leqslant C_\varepsilon \lambda^\varepsilon \mu.$$

By partial summation, the left side of (1.14) is estimated

$$C_\varepsilon \lambda^\varepsilon \Bigl(\lambda^{\frac{\beta}{2}} + \sum_{\mu \leqslant \lambda^{1/2}} \mu\mu^{\beta-1}\Bigr) \leqslant C_\varepsilon \lambda^{\frac{\beta+1}{2}+\varepsilon}.$$

This proves the proposition. □

Going back to (4.9), estimates (4.8) and (1.13) imply

$$\max_{\tau \approx 1/(qN)} \bigl|S_u(q,\tau)\bigr| \leqslant C_\varepsilon \lambda^{\frac{n}{4}} q^{\frac{1}{2}+\varepsilon} (\lambda, q_1)^{\frac{1}{2}} 2^{\frac{r}{2}} \tag{4.16}$$

with a constant $C_\varepsilon > 0$ independent of u and ξ. Substituting this into (4.2) and using (1.14) for $\beta = 1/2$ one estimates the error term by

$$\bigl|E_1(\lambda)\bigr| \leqslant C_\varepsilon \lambda^{\frac{n-1}{4}+\varepsilon}. \tag{4.17}$$

Next, we do a number of transformations on the main term in (4.1) to arrive to the asymptotic formula (1.8) and estimate the error obtained in each step. These are similar to those in [MSW, Proposition 4.1], except here we do not have to deal with maximal functions. The better error term of the order of $\lambda^{\frac{n-1}{4}+\varepsilon}$ comes from inequality (1.13).

The main term in (4.1) takes the form

$$\begin{aligned} &\sum_{q \leqslant N} \int_{-1/(qN)}^{1/(qN)} e^{-2\pi i \lambda \tau} S_0(q,\tau)\, d\tau \\ &\quad = \sum_{q \leqslant N} \sum_{l \in \mathbb{Z}^n} K(q,l,\lambda;0) \int_{-1/(qN)}^{1/(qN)} e^{-2\pi i \lambda \tau} \tilde{h}_{\tau,\delta}(\xi - l/q). \end{aligned} \tag{4.18}$$

First, one inserts the cut-off functions $\psi(q\xi - l)$ into the l-sum in (4.18). Note that $|q\xi - l| \geqslant 1/8$ on the support of $1 - \psi(q\xi - l)$ thus by (4.8) one has

$$\sum_{l \in \mathbf{Z}^n} \bigl(1 - \psi(q\xi - l)\bigr) \bigl|\tilde{s}_{\tau,\delta}(\xi - l/q)\bigr| \leqslant C\bigl(\tau^2 + \delta^2\bigr)^{-\frac{n}{4}} e^{\overline{\frac{c\delta}{q^2(\tau^2+\delta^2)}}} \tag{4.19}$$

with some absolute constants $C, c > 0$. Using the fact that $e^{-u} \leqslant C u^{-n/4}$ for $u = \tau^2 + \delta^2$ one estimates the left side by $C\lambda^{n/4} q^{n/2}$. Thus the total error accumulated by inserting the cut-off functions $\psi(q\xi - l)$ in (4.19) is bounded by

$$\bigl|E_2(\lambda)\bigr| \leqslant C_\varepsilon \lambda^{\frac{n}{4}-\frac{1}{2}} \sum_{q \leqslant N} q^{-\frac{1}{2}+\varepsilon} (\lambda, q)^{\frac{1}{2}} \leqslant C_\varepsilon \lambda^{\frac{n-1}{4}+\varepsilon} \tag{4.20}$$

and the main term takes the form

$$\sum_{q\leqslant N}\sum_{l\in\mathbb{Z}^n} K(q,l,\lambda;0)\psi(q\xi-l)\int_{-1/(qN)}^{1/(qN)} e^{-2\pi i\lambda\tau}\tilde{h}_{\tau,\delta}(\xi-l/q). \tag{4.21}$$

Next, the integration is extended to the whole real line. Note that now there is at most one non-zero term in the l-sum, and for $|\tau|\geqslant\frac{1}{qN}\geqslant\delta$ one has $|\tilde{s}_{\tau,\delta}(\xi-l/q)|\leqslant C\tau^{-n/2}$. The total error obtained in (4.21) by extending the integration is

$$\left|E_3(\lambda)\right|\leqslant C_\varepsilon\sum_{q\leqslant N} q^{-\frac{1}{2}+\varepsilon}(\lambda,q)^{\frac{1}{2}}\int_{|\tau|\geqslant 1/(qN)}\tau^{-\frac{n}{2}}\,d\tau\leqslant C_\varepsilon\lambda^{\frac{n-1}{4}+\varepsilon}. \tag{4.22}$$

Finally by identifying the integrals (see [MSW, Lemma 6.1])

$$\int_{\mathbb{R}} e^{-2\pi i\lambda\tau}\tilde{h}_{\tau,\delta}(\xi)\,d\tau=\gamma_n'\lambda^{\frac{n}{2}-1}\tilde{\sigma}\left(\lambda^{\frac{1}{2}}\xi\right) \tag{4.23}$$

one arrives at the asymptotic formula (1.8) with error term $\mathcal{E}_\lambda(\xi)=E_1(\lambda)+E_2(\lambda)+E_3(\lambda)$. Estimate (1.9) follows from (4.17), (4.20) and (4.22). This proves Lemma 1.

5. Level surfaces of polynomials

The aim of this section is to emphasize that estimates for the discrepancy with respect to caps of diophantine directions generalize to level surfaces of polynomials of higher degree. The proof proceeds exactly as in Section 3, using the asymptotic formula (0.6) proved in [M1], to be described below.

Let $p(m_1,\ldots,m_n)$ be a non-singular, positive, homogeneous form of degree d, and assume that $n>(d-1)2^d$. Then there exists an $\eta'>0$ depending only on n and d, such that

$$\hat{\omega}_\lambda(\xi)=\gamma_{p,n}\lambda^{\frac{n}{d}-1}\sum_{q\leqslant\lambda^{1/2}} m_{q,\lambda}(\xi)+\mathcal{E}_\lambda(\xi), \tag{5.1}$$

where

$$\left|\mathcal{E}_\lambda(\xi)\right|\leqslant C_p\lambda^{\frac{n}{d}-1-\eta'} \tag{5.2}$$

holds uniformly in ξ. Moreover

$$m_{q,\lambda}(\xi)=\sum_{l\in\mathbb{Z}^n} K_p(q,l,\lambda)\psi(q\xi-l)\tilde{\sigma}_p\left(\lambda^{\frac{1}{d}}(\xi-l/q)\right), \tag{5.3}$$

$$K_p(q,l,\lambda)=q^{-n}\sum_{(a,q)=1}\sum_{s\in(\mathbb{Z}/q\mathbb{Z})^n} e^{2\pi i\frac{a(p(s)-\lambda)+s\cdot l}{q}} \tag{5.4}$$

and $\tilde{\sigma}_p$ denotes the Fourier transform of the measure σ_p on the unit level surface S_p, defined in the introduction.

We invoke the basic estimates [M1, (1.5) and (1.13)]

$$\left|\tilde{\sigma}_p(\xi)\right| \leqslant C\left(1+|\xi|\right)^{-\frac{\kappa}{d-1}+1+\varepsilon}, \tag{5.5}$$

$$\left|K_p(q,l,\lambda)\right| \leqslant C_\varepsilon q^{-\frac{\kappa}{d-1}+1+\varepsilon}. \tag{5.6}$$

Here $\kappa = n/2^{d-1}$ as we assume that p is non-singular, that is the singular variety $V_p = \{0\}$. To simplify the computations, let $\tau = (\frac{\kappa}{d-1} - 2)/2 > 0$. Assuming $\varepsilon < \tau$, the exponents $-\frac{\kappa}{d-1} + 1 + \varepsilon$ in (5.5) and (5.6) can be replaced by $-\tau - 1$.

We turn to the proof of Theorem 2. Note that the smoothing estimate (3.3) holds in this case as well. Define $I_{q,\lambda}$ and E_λ as in (3.6) and (3.7), with the only change that $\lambda^{1/2}$ is replaced by $\lambda^{1/d}$. By (5.2) one has

$$|E_\lambda| \leqslant C_p \lambda^{\frac{n}{d}-1-\eta'}. \tag{5.7}$$

We decompose the integral $I_{q,\lambda}$ as in (3.9), and note that for $|t| < 1/8q$

$$m_{q,\lambda}(t\xi) = K_p(q,0,\lambda)\tilde{\sigma}_p\left(\lambda^{\frac{1}{d}} t\xi\right).$$

Using (5.2) and (5.4) and arguing as in Proposition 5, one obtains

$$\begin{aligned} &\left|\gamma_{n,p}\lambda^{\frac{n}{d}-1} \sum_{q\leqslant\lambda^{1/d}} I^1_{q,\lambda} - N_\lambda \int\limits_{S_p} \phi_{a,\delta}(x\cdot\xi)\, d\sigma_p(x)\right| \\ &\quad\leqslant C\lambda^{\frac{n}{d}-1}\left(\lambda^{-\eta'} + \sum_{q\leqslant\lambda^{1/d}} \left(\lambda^{\frac{1}{d}}/q\right)^{-\tau} q^{-\tau-1}\right) \leqslant C\lambda^{\frac{n}{d}-1-\eta} \end{aligned} \tag{5.8}$$

with say $\eta = \min(\eta', \tau/d)/2$.

By making a change of variables $t := tq$, it follows for (5.5) and (5.6)

$$\left|I^2_{q,\lambda}\right| \leqslant C\lambda^{-\frac{\tau}{d}} q^{-1} J_\lambda, \tag{5.9}$$

where

$$J_\lambda = \int\limits_{|t|\geqslant 1/8} \left|\hat{\phi}_{a,\delta}\left(t\lambda^{\frac{1}{d}}/q\right)\right| \|t\xi\|^{-\tau-1}\, dt. \tag{5.10}$$

Arguing as in (3.19) and (3.20) gives: $|J_\lambda| \leqslant C_\varepsilon(\lambda^{1/d}/q)^{-1}\lambda^\varepsilon$ (note that $0 < \tau + 1 < n - 1$). Thus

$$\sum_{q\leqslant\lambda^{1/d}} \left|I^2_{q,\lambda}\right| \leqslant C_\varepsilon \lambda^{-\frac{\tau}{d}+\varepsilon}. \tag{5.11}$$

Theorem 2 follows from (5.7), (5.8) and (5.11).

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